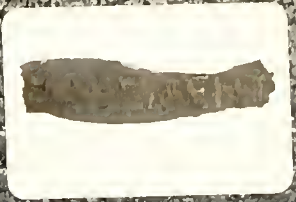

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CONTINUOUSLY DATED COMMODITIES, INDIVIDUAL PRICE ADJUSTMENT, AND NON-TÂTONNEMENT WITH PRODUCTION AND CONSUMPTION

by

Franklin M. Fisher

Number 168 November 1975

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1. Introduction: The Problem of Dated Commodities

In my earlier paper [7], I presented a non-tâtonnement model in which production and consumption are allowed to take place out of equilibrium and proved a stability theorem. That model may have represented some advance over earlier work in which trading, but not other economic activity, was allowed to take place out of equilibrium, but its construction brought to the fore a number of serious difficulties which are implicitly present in all general equilibrium stability analyses but which must be explicitly faced when production and consumption are allowed.[2] The present paper offers one way to handle one of the more awkward of those difficulties, the problem of dated commodities.

The dated commodities problem can be stated as follows. If, as is both traditional and highly natural, we distinguish between the same commodity at different dates, then we must consider the question of whether or not some commodity dates are passed before equilibrium is reached. When dealing with tâtonnement processes, this is not a problem, since adjustment to equilibrium occurs before anything interesting happens anyway; in dealing with non-tâtonnement trading processes, that is, processes in which titles to goods are traded out of equilibrium but in which no other activity takes place until the music stops, it is also possible (though awkward) to ignore the question by assuming that adjustment somehow takes place outside of real time. When production and consumption take place out of equilibrium, however, it must be the case that commodity dates are passed in the process; future pie is pie in the sky and only currently dated commodities can be consumed.

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[1] This research was supported by National Science Foundation Grant GS 43185. I am grateful to Hal Varian for many helpful conversations but retain responsibility for error.

[2] Fisher [8] presents a relatively non-technical discussion of such problems as well as a survey of the field. Earlier surveys are given in Negishi [15] and Arrow and Hahn [1].

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Now, if commodity dates are passed during the adjustment process, then individuals may be suddenly surprised, given the assumptions about expectations that are usually made in these analyses. While there can perfectly well be futures markets, there cannot be "pasts" markets, and trading in a commodity whose date is passed will be suddenly suspended. Since it is common to suppose that individuals stupidly fail to realize that the economy is not in equilibrium and in any case believe that they personally will be able to complete their desired transactions at the current prices, there will generally be individuals who suddenly realize that they are stuck with whatever stock of a particular dated commodity they happen to hold since it can no longer be traded. Such individuals must reoptimize their behavior subject to the suddenly imposed constraint that they can no longer trade the particular commodity involved. Since the constraint comes as a sudden surprise, this will generally involve a discontinuity in their excess demand functions for other commodities. This means a violation of Lipschitz Conditions and of the whole underpinning for assuming a continuous solution to the underlying differential equations of the model. It also violates the conditions involved in the usual proofs of the stability of those equations. This is clearly quite troublesome.

It may be thought, however, that the trouble is more illusory than real. After all, as midnight on December 31 approaches, I typically do not lose any sleep over the fact that I will not be able to buy this year's toothpaste, because I expect to be able to buy next year's toothpaste and, at midnight, these are perfect substitutes. The catch is that such an argument implicitly supposes that, as midnight approaches, I become progressively indifferent to which of the two commodities I buy because their prices come together. This is an equilibrium consideration, however, and the usual price adjustment processes do not, in general, lead to this result if the two differently dated toothpastes are treated as two different commodities. Out of equilibrium, therefore, I can perfectly well be unpleasantly surprised at midnight by the discovery that I must now purchase a suddenly higher-priced commodity if I wish to brush my teeth.

There are a number of possible ways in which one might attempt to get round this problem. The first of these is to assume it away, or, what comes to the same thing, to assume that commodities are not dated. This is the approach taken in [7]. Not only is it clearly unsatisfactory on its own lack of grounds, however, but it also exacerbates the difficulties associated with another hidden assumption which comes to the fore when serious economic activity is allowed outside of equilibrium, an assumption which I have called the "Present Action Postulate."

Essentially, the Present Action Postulate requires that if an individual (household or firm) has an unsatisfied excess demand for some commodity, then he must attempt to exercise that demand immediately rather than hanging back and planning to exercise it later. Some such assumption is clearly both implicit and crucial in any model in which prices are to be driven by excess demands, since prices can only be influenced by excess demands which are acted upon; nevertheless, it is very strong.[1] When commodities are not dated, however, the Present Action Postulate is even stronger, for it requires, for example, that someone who expects to need toothpaste in 1985 must immediately begin to attempt to acquire it. When commodities are not dated, this means that he must act on the spot market for toothpaste since there is no futures market. The problem is even worse for firms if commodities are undated. They must now begin to sell any commodity which they expect eventually to produce, even if production takes a long time and even if they wish first to use such a commodity as an input. Obviously, this would not be nearly so troublesome if commodities were distinguished by dates and it is really not very satisfactory to assume that they are not so distinguished.

A second possible approach to the dated commodities problem is not to ignore possible discontinuities in prices, but rather to suppose that nobody is surprised by them. One can proceed by assuming that, as the date on a particular commodity approaches, it becomes harder and harder to trade it so that there are utility or resource losses involved in finding someone to trade with. Such an approach can probably deal with the problem, but it seems relatively ad hoc without an underlying model of individual behavior showing why such continuously increasing difficulty of trade should take place. Merely stating that individuals realize that they are more and more likely to get caught holding the bag as the date for suspension of trading approaches imports into the model in a special place a realization of disequilibrium that is conspicuously absent from it elsewhere. Nevertheless, it would probably be worthwhile working out such a model in full detail.[1]

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[1] For a fuller discussion, see [7] and [8].

If one is going to allow consciousness of disequilibrium, however, it would obviously be desirable to do so properly and to allow for arbitrage as the principal device which keeps discontinuities from occurring when a particular date is passed. Unfortunately, when one has said this one has not really made a great deal of progress. Allowing arbitrage of the sort involved here is no different than allowing there to be profit-maximizing firms who have access to a technology enabling them to turn present goods into future ones. Such firms can already exist in the model.[2] The problem lies in assuring that their arbitrage will wipe out any discontinuities in price by the crucial dates.

Clearly, one cannot hope to assure this without dealing with what is really at the heart of the problem, the process by which prices adjust. Unfortunately, despite its obvious central importance, we lack an adequate understanding of how competitive prices change. The reason for this is not hard to find. Micro-economic competitive theory is preeminently an equilibrium theory in which all participants take prices as given. Such a theory will find it difficult to explain how prices ever change.[1] The device of the fictitious auctioneer who changes prices in accord with total excess demand has remained behind most price adjustment processes, in non-tâtonnement as well as tâtonnement models. When one seeks, as in the present problem, to discover more structure in the price adjustment process, that device no longer proves very useful, for it is hard to gain much insight into adjustment rules based on nobody's behavior.

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[1] Fisher [8A] shows one way in which quantity constraints on trading taken into account by agents can be incorporated into Hahn Process models. It might be possible to adapt that analysis to have upper and lower bounds on trading on any dated commodity come together as its date is reached.

[2] It is true that the model in [7], as in the present paper, cannot permit constant returns and requires stocks to remain bounded. In these respects it might be claimed that the firms therein are prevented from fully performing the necessary arbitrage. Aside from the issue of whether such assumptions are unrealistic, given the limited capacities of storehouses and individuals and the bounded creditworthiness of speculators, the point being made in the text does not depend on such assumptions.

As soon as one begins to think about individuals as setting their own prices, however, a way around the dated commodity problem becomes apparent. If the same household or firm is selling both this year's and next year's toothpaste and posting the prices for both of them, then it will be to that entity's interest to avoid a discontinuity as the date passes. It will realize that if it allows discontinuity, then, at the crucial moment, it will be posting what in effect are two different prices for the same thing. It cannot hope to sell at the higher of those prices, since buyers can postpone or advance their purchases by a small amount of time to take advantage of the lower one, and, in effect, it will be encouraging arbitrage at its own expense.

Note that this need not involve the assumption that all commodities are storable for short periods. What is required is that commodities distinguished only by their dates become perfect substitutes as the dates coincide. This can hold for both "stock" commodities which are storable and for "flow" commodities like labor services. What is involved in the latter case is not the question of whether working this year and working next year are perfect substitutes but rather the question of whether working now and working a little later become perfect substitutes as the postponement involved goes to zero.

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[1] This was pointed out by Koopmans [13] among others.

Moreover, if one considers individual market participants as having responsibility for posting their own prices for given commodities, both present and future, one can deal with an awkwardness closely related to that which we have been discussing. It is somewhat unnatural in models which work in continuous time to have commodities which are dated and distinguished only by discrete dates. There may in fact be good reasons of convenience and transaction costs why futures markets arise only for discrete dates[1] but they probably ought to be dealt with explicitly rather than being assumed into the basic nature of the commodities. (In fact, we somewhat accomodate such considerations below.) In any case, it seems somewhat more natural in such models to work with continuously dated commodities. One can then think of prices as price profiles extending over future time. Clearly, if one could guarantee that such price profiles were always continuous in the commodity dates, the problem which we have been discussing could not arise.[2] If prices for different dates are set by different people using different rules, such continuity cannot be assured; if,on the other hand, we think of individuals as setting such profiles -- the terms on which they stand ready to deliver (buy) at different times -- then such continuity, as above, will be a natural consequence of the self-interest of the price-setters and the fact that two commodities identical except for dates become perfect substitutes as the dates coincide.

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[1] On this, see, for example, Foley [9] and Hahn [10].

[2] Note, however, that this is stronger than necessary. The really important thing is to assure continuity of the price profile at the current time, that is, as futures prices become spot prices. Continuity at future dates is convenient but not essential.

The present paper, then, extends the work of [7] by building such a model with continuously dated commodities. It also indicates briefly how work of my earlier paper [4] on individual price adjustment can be extended to a model involving firms, production, and consumption, from one of pure exchange. Unfortunately, models of individual price adjustment bring with them their own problems, to a discussion of which I now turn.

2. Individual Price Adjustment

If one seeks to have a model in which real participants set their own prices, then one is more or less forced to deal with cases of less than complete information or mobility. If customers always know about the lowest-priced seller and can deal with him at no special cost, then no price dispersion can ever exist. In such circumstances, it is hard to see why any seller would or could raise prices above the minimum then existing. This makes it natural to deal with a model in which customers search over sellers to find the lowest price.[1]

Unfortunately, such lack of perfect information on the part of customers gives sellers some monopoly power. Most of the growing literature on search processes takes such power into account in one way or another[1], but its existence is awkward for models of the stability of competitive equilibrium, such as the present one. The problem can be put as follows. If sellers are not allowed to take advantage of their monopoly power, then they must be supposed to behave in a moderately foolish way. On the other hand, if they are allowed to optimize while taking that power into account, then it is far from clear that the resulting model will converge to competitive equilibrium, if it converges at all.

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[1] Throughout this paper, I shall generally speak as though sellers always set prices and buyers do the searching. This is not a necessary assumption.

An obvious question, of course, is that of why we are particularly interested in establishing convergence to competitive equilibrium in such models rather than in analyzing them for their own implications. While such analysis is of course of great interest, it is also important to discover whether reasonable price-adjustment models can be made to converge to competitive equilibrium, for most of what we have to say about microeconomics depends at least implicitly on the assumption that competitive equilibrium exists and is attainable.[2]

It would thus be desirable to build a model in which the monopoly power referred to above is taken into account and competitive equilibrium nonetheless achieved. For a single market, I did this in [5], although the assumptions that appeared to be required were quite onerous. I do not know whether it can be done for a general equilibrium model.

Accordingly, in the present paper, whose principal object is somewhat different, I have implicitly taken the far easier, if less appealing course followed in [3] and [4] of having price setters act without full consciousness of their monopoly power and, indeed, without full consciousness of disequilibrium. This is, of course, fully consonant with the traditional development of stability analysis in which participants constantly behave as though they were in equilibrium. That does not make it more satisfactory.

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[1] See Rothschild [16] for a survey.

[2] The same problem exists at the partial as at the general level. For a fuller discussion of these issues, see [8].

In the present model, then, there is, for each commodity, a group of recognized price setters or dealers. It is natural to think of these as firms, but this is not necessary. Each dealer sets a continuous price profile for his commodity, giving the prices at which he will sell his commodity for delivery at any future time. He also decides how much he is willing to deliver at such prices. The dealer behaves in his price-setting as though he were hunting for a competitive equilibrium in the sense that, if he finds he cannot sell all that he wished, he assumes he has set too high a price and reduces his price; similarly, if he must turn away unsatisfied customers, he assumes he has set too low a price and raises his price. As the price profile he sets is continuous and as we shall assume (as is natural) that the resulting excess demand profiles are also continuous in the delivery date, this can be assumed to result in a continuous alteration of the price profiles.

The dealer's decision as to how much to offer for sale at each future date is based on optimizing calculations given all the prices. If there are disutilities or costs associated with delivery during some future time intervals, then he may very well decide to offer nothing during those intervals unless the price is sufficiently high. Hence, while we assume that he sets a continuous price profile defined over all future delivery dates, this does not require that the corresponding continuum of futures markets be active in any but a nominal sense. Presumably, for high enough prices, the difficulties surrounding delivery during particular intervals will be overcome.

There is considerable payoff to using such a model of individual price adjustment. We have already seen how it can naturally lead to a way around the dated commodity problem. Further, as in [4], it makes the central assumption of the Hahn Process -- that, after trade, there cannot be both unsatisfied demanders and unsatisfied suppliers of a given commodity -- almost compelled if we distinguish commodities by dealers so that the dealer is the only one on a particular side of his market. Finally, also as in [4], it makes the No Bankruptcy assumption of the Hahn Process more palatable. As it is the Hahn Process which seems natural to use in models with production and consumption (as observed in [7] and described in the next section), these are reasons for incorporating individual price adjustment explicitly.

On the other hand, such explicit incorporation does burden both the exposition and the notation. Accordingly, I have avoided the explicit introduction of such considerations in the presentation of the model below, introducing them very briefly at the end, in the hope that this will make the analysis easier to follow. This also means that the reader can take the model before such introduction as an analysis of dated commodities and stability on the unsupported assumption that continuous price profiles are set and continuously (if mysteriously) altered.

3. The Hahn Process and Money

As already mentioned, the basic adjustment process used is the Hahn Process[1] which assumes that there cannot be simultaneously (after trade) unsatisfied demanders and unsatisfied buyers of the same commodity. Such an assumption really has two parts. The first of these is that markets are sufficiently well organized and information flows good enough that buyers and sellers of the same commodity can find each other. As already remarked, in models in which commodities are distinguished by the identity of the dealer involved, such an assumption is almost compelled.

It is not enough for prospective traders to meet, however. In order that trade be consummated, the prospective buyer must have something of value to offer the prospective seller.[2] This problem was first satisfactorily formalized (although certainly not resolved) in Arrow and Hahn [1] where it is assumed that trades are only in exchange for a particular commodity, called "money." Only purchase attempts backed up with money, so-called "active excess demands", can influence prices; desired or "target" excess demands do not, except insofar as they give rise to active ones. Target and active excess demands coincide when negative, since offers to sell do not have to be backed up with money, and are of the same sign when target excess demands are positive, so long as the individual in question has a positive stock of money to ration out over the commodities he wishes to purchase. In this setup, the Hahn Process assumption requires, in effect, that all money stocks remain positive.

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[1] So named by Negishi [15]. The basic paper is Hahn and Negishi [7]. Further developments can be found in Arrow and Hahn [1, Ch. XIII] and Fisher [4], [6], [7], and [8A].

[2] For a more extended discussion, see [8].

Unfortunately, it is hard to derive such a No-Bankruptcy condition from more fundamental considerations.[1] In a model with individual price adjustment, however, such a condition is relatively more palatable. This is because, with individual control over price, a dealer who sees his money stock running out and who has set a high price can quickly reduce that price so as to attract customers. This goes some way toward justifying the No-Bankruptcy condition.[2]

I shall implicitly adopt the Arrow-Hahn version of the Hahn Process as just described; in the present paper, however, there is no need to do so explicitly. This is because we shall work exclusively in continuous time. Since we shall have to assume money stocks positive anyway[3] and since target excess demands are demands for instantaneous flows, expenditures on target excess demands can never exhaust money stocks. Hence, there is no need to distinguish explicitly between target and active excess demands, and, indeed, no need to have an explicit notation distinguishing money from other goods. Since the consequent saving in notation and expositional simplicity is non-trivial, we shall not make such considerations explicit in the formal model, although they form part of the background which should be borne in mind.

The basic consequence of the Hahn Process assumption is that any non-zero individual excess demand is of the same sign as aggregate excess demand for the same commodity. Since prices adjust in the direction indicated by aggregate excess demands, this means that every participant in the model finds that the things he wishes to buy are becoming more expensive, while the things he wishes to sell are becoming cheaper. As a result, the profits which firms expect to make if they complete their transactions (target profits) are declining out of equilibrium; similarly, the utilities which households expect to achieve if they can complete their transactions (target utilities) are declining out of equilibrium, both because of their direct market experiences and because they expect to receive progressively less and less in future dividends from firms.

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[1] See Arrow and Hahn [1] for discussion.

[2] See [4].

[3] Money stocks would not have to be positive if we could ensure that some sale for each participant was always actually consummated or, for households, that firms in which they hold shares always managed to realize some profits and pay dividends; there seems no reason to assume this, however.

Such declining hopes would take place even if all activities were reversible. When consumption and production are also taking place, that decline is reinforced. This is because the undertaking of irreversible production or consumption activities which are optimal on the basis of preexisting prices can only result in a situation when prices change in which target profits or utilities are no greater than would have been the case if such activities could be reversed and reoptimized given the new prices.

Thus target utilities decline out of equilibrium and the sum of target utilities can be used as a Lyapounov function. This is the core of the stability proof. It seems natural if we think of the process as one in which inconsistent profit and utility expectations are revised downward until equilibrium is reached, equilibrium being a point at which plans based on such expectations are all mutually consistent.

4. Topological Issues

Before proceeding with the formal model, we digress to discuss the issues surrounding the choice of space and norm with which to work. As is natural to do, we shall take consumption, inputs, outputs, and the like to be (usually continuous) profiles defined on $[0,\infty)$ and lying in some normed linear space. Prices will then be taken to lie in the normed dual of that space. At the level of generality at which we shall be working, however, no direct use will be made in the proofs of the particular properties of the chosen space and its normed dual.[1] Hence there is little reason to restrict ourselves to a particular space. Rather, all results should be read as applying to any choice such that our other assumptions are satisfied.

That is not to say, however, that the choice of a normed linear space is a matter of no consequence, for, of course, the interpretation of assumptions and results depends on what space is chosen. In the present paper, this matters in three areas, the third of which is somewhat subtle.

4.1. Differentiability and Continuity

Utility and production functionals are assumed Fréchet differentiable in their arguments. Moreover, the optimization problems faced by households and firms are assumed sufficiently regular that their solutions are weak* continuous in the prices. What such assumptions mean depends on the choice of a normed space in which to work. On the other hand, the plausibility of such assumptions seems about the same for a large number of such choices.

4.2. Boundedness

It is important to the proof of stability that prices lie in a compact set. The natural way to go about this is to work with prices in the weak* topology, show or assume weak* closure and boundedness and apply the Alaoglu Theorem[1]. As it turns out, the difficult issue here is weak* boundedness.

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[1] Although particular choices, together with rather restrictive assumptions would lead to proofs of boundedness, as discussed below.

Now, in fact, boundedness of prices, while plausibly assumed, is at best awkward to prove in models of this sort even with a finite number of commodities.[2] One way to do it is to assume that the price-adjustment functions are bounded above by a ray through the origin and that there is excess demand for the numéraire if its price becomes relatively low enough. One can then use Walras' Law to show that the sum of squares of prices is bounded. Alternatively, if one does not wish to restrict the price-adjustment functions, one can establish boundedness of prices in the finite-dimensional case by restricting demands so that if any group of prices gets relatively high enough excess demand for the highest-priced commodity is nonpositive. One can then show that the maximum price is bounded.

Such methods will also carry over to the present case; they are less useful here, however, because they allow the proof of boundedness only for particular choices of norm, since not all norms are equivalent in this respect as in the finite-dimensional case. Thus, the proof assuming a restriction on the price-adjustment functions which shows boundedness of the sum of squares can here be adapted to show boundedness of prices, provided that the appropriate norm is the square root of the integral of squared price as it would be if prices lay in L_2. Similarly, the proof which proceeds by restricting the demand functions to show boundedness of the maximum price can here be adapted (given continuity of prices in commodity dates) to show boundedness of prices, provided that the appropriate norm is the essential supremum, as it would be if prices lay in L_∞. Unfortunately, such results do not appear readily to extend to other choices of norm (although, given continuity of prices in commodity dates the first method also implies boundedness for the essential supremum).

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[1] See, e.g., Luenberger [14, p. 128].

[2] See Fisher [7] for details.

One might therefore suppose that the sensible thing to do is to choose the original commodity space as L_2 or L_1, so as to have prices turn out to lie in L_2 or L_∞, respectively. While such choices are possible ones, however, one gains from them only the ability to prove boundedness of prices using assumptions that are pretty restrictive and not obviously much more plausible than the direct assumption of boundedness itself.[1] Moreover, it turns out that one loses a good deal in the interpretation of the results with these particular choices, for they suffer from what I shall call the "vanishing norm property," to a discussion of which I now turn.

4.3. Vanishing Norm

In a notation restricted to the present section, let q be a quantity profile (consumption, input, etc.) defined over $[0,\infty)$ and lying in a normed linear space, X. (For present purposes $q(\theta)$, $\theta\varepsilon[0,\infty)$, can be taken to be a scalar, to simplify matters.) For any $\lambda > 0$, define q_λ in X as:

---

[1] Since the only purpose of assuming or proving boundedness is to obtain weak* compactness, and since weak* convergence of prices is not the strongest result one could wish, one might ask why we do not go ahead and simply assume compactness for prices in some stronger topology. The answer is that it is hard to see why such an assumption is particularly plausible unless it is equivalent to closure and boundedness.

(4.1) $$q_\lambda(\theta) = \begin{cases} 0 \text{ for } 0 \leq \theta \leq \lambda . \\ q(\theta) \text{ for } \theta > \lambda . \end{cases}$$[1]

Definition 4.1: X will be said to have the vanishing norm property if and only if for every q in X, $\text{Lim}_{\lambda \to \infty}$ norm $(q_\lambda) = 0$.

Heuristically, a space with the vanishing norm property is one in which differences between profiles which occur far in the future are given less and less weight. Examples are easy to come by. All the L_p spaces possess the vanishing norm property for $1 \leq p < \infty$, although L_∞ does not. In particular, L_1 and L_2, the two spaces which turned up in the discussion of boundedness, have it.

Now the reason that the vanishing norm property is an inconvenience is related to the fact that economic activity is occurring during the adjustment process. Hence, just through the passage of time, some prices -- those for commodities whose dates have been passed -- become fixed. One hardly wants to have this fact alone mean that prices converge, but this is essentially what happens if quantity differences later in time get less and less weight.[2]

More precisely, let X* be the normed dual of X. Let $\bar{p}$ in X* be the actual time path of spot prices, so that for any θ, the value of $\bar{p}(\theta)$ is the closing price for the single commodity dated θ which actually obtained at θ. For any $\lambda > 0$, let p_λ in X* be a profile such that $p_\lambda(\theta) = \bar{p}(\theta)$ for $0 \leq \theta \leq \lambda$. We can think of p_λ as a profile which shows actual closing prices for commodities dated up to λ and the futures prices obtaining at λ for commodities dated after λ. It is now easy to prove:

---

[1] It is assumed that q_λ is in X. If the possible discontinuity at $\theta=\lambda$ prevents this, we could easily overcome it by having $q_\lambda(\theta)$ go continuously from 0 at λ to the value of $q(\theta)$ at some θ slightly greater than λ.

[2] Note, however, that such heuristics can be a little misleading. It turns out that the crucial issue as to trivial convergence of prices is whether the commodity space, X, has the vanishing norm property. It does not matter whether the price space (the normed dual of X) does.

Lemma 4.1: If X has the vanishing norm property, then, provided that $\bar{p}$ and the p_λ lie in a bounded set, the sequence of the p_λ converges weak* to $\bar{p}$ as λ goes to ∞.

Proof: Since p_λ and $\bar{p}$ lie in a bounded set,

$$(4.2) \qquad \left| \int_0^\infty (p_\lambda(\theta) - \bar{p}(\theta))\, q(\theta)\, d\theta \right| = \left| \int_\lambda^\infty (p_\lambda(\theta) - \bar{p}(\theta))\, q(\theta)\, d\theta \right|$$

$$= \left| \int_0^\infty (p_\lambda(\theta) - \bar{p}(\theta))\, q_\lambda\, (\theta)\, d\theta \right| \leq \text{norm}\ (p_\lambda - \bar{p})\ \text{norm}\ (q_\lambda)$$

$$\leq K\ \text{norm}\ (q_\lambda)\ ,$$

where K is some finite positive scalar independent of λ. Since X has the vanishing norm property, the extreme right-hand side of (4.2) approaches zero as λ goes to infinity, proving the lemma.

Note that such a result is far stronger than that of the Alaoglu Theorem which would merely show (given closure as well as boundedness) that the sequence of the p_λ has a convergent subsequence.

It is obviously unattractive to have prices converge weak* just because of boundedness and the fact that one has chosen a particular norm.

There is nothing wrong with such a choice, however, if one is willing to put up with this and, although the weak* convergence of prices (and the consequent convergence of quantities assumed weak* continuous in the prices) becomes trivial, the fact that the limit involved is a competitive equilibrium is not. Nevertheless, it is clearly undesirable to produce such triviality just by choosing the way in which one is going to measure things, particularly because the same result can be achieved in a far from trivial way.

We now proceed to the formal model.

5. Behavior of Households

There are n commodities, of which the nth called "money" is the numéraire. Each commodity is dated continuously. Commodity dates are most often denoted by θ. As of time t, the vector of price profiles for the n commodities is denoted by $p(\theta,t)$. Clearly, the value of a particular element of $p(\theta,t)$ for a particular $\theta \geq t$ is a futures price for a particular commodity. For technical reasons, however, it is convenient to think of $p(\theta,t)$ as defined for all θ in $[0,\infty)$, where 0 is some arbitrary starting date, even though trading is not permitted for $\theta < t$. For such past dates, the prices involved remain fixed at levels corresponding to the spot prices which obtained at θ. Each element of $p(\theta,t)$ is a continuous function.

For a number of reasons, it is inconvenient explicitly to allow money to be distinguished by date so as to have different prices for different times in the future. We assume that money, whenever dated, has price equal to unity, so that the nth component of $p(\theta,t)$ is a function always equal to unity. There is no market on which future money trades for present money at par, but money can always be stored. The consequent elimination of credit and interest, however, is only apparent. We may think of the n currently dated commodities as including commodities called "bonds" with the following properties. A particular bond is characterized by a specific maturity period. The consumption of such a bond adds nothing to utility but included in the technological opportunities open to firms and the storage activities open to households is the ability to place such bonds in storage at time t and to have each unit so stored turn into one unit of money at the end of the maturity period. Bonds can be negatively as well as positively stored, so that trading in bonds exchanges future for present money at a price to be determined in the usual way. (For reasons of boundedness some credit limit must be imposed on the ability of any one entity to engage in such transactions, however.)

I shall not make any of this notationally explicit. Indeed, it will simplify the notation greatly to adopt a notation throughout which treats money in a way parallel to the other commodities as is already done in $p(\theta,t)$. In order to do this, however, one must interpret integrals such as $\int_0^\infty m(\theta,t)d\theta$, where $m(\theta,t)$ is some profile of money, as equal to the amount of spot money involved, even though $m(\theta,t) = 0$ for $\theta > t$. This occurs in particular for integrals showing the amount of money paid in purchase transactions and for profits and dividends. I apologize for the slight awkwardness involved in making the monetary features largely implicit, but it really does make the exposition much simpler.[1]

Households will later be subscripted h, but the subscript will be omitted until needed much later on. Each household has an n-vector of consumption profiles $c(\theta,t)$, where θ is in $[0,\infty)$. For $\theta \leq t$, the corresponding values of $c(\theta,t)$ represent actual consumption, while for $\theta \geq t$, they represent consumption planned at t to take place at θ. (It is assumed that these coincide for $\theta = t$, which involves assuming that households are not so mistaken about their possessions and transactions that they plan to consume out of current transactions which do not materialize. I shall return to this below.)

The household attempts to vary the parts of $c(\theta,t)$ which are not already fixed to maximize a utility functional $U(c(\theta,t))$. Note that, since consumption does take place during the adjustment process, the situation is one in which the thing being currently maximized is altered by past consumption activities. $U(.)$ is assumed Fréchet differentiable and strictly quasi-concave.[2]

In maximizing $U(.)$, the household of course is constrained in various ways. We begin by considering its marketing activities.

We denote by $r(\theta,v,t)$ an n-vector of purchase profiles. Here, a particular value of an element of $r(\theta,v,t)$ denotes the amount of a commodity dated θ which the household, as of time t, plans to purchase at time v. Purchases are measured positively.

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[1] Some (though not all) of the simplification can be seen on comparing the notation of the present paper with that of [7].

[2] It is notationally convenient to have household consumption functions defined over all commodities even though some of them such as intermediate goods or bonds may not affect utility directly. Strict quasi-concavity is to be interpreted as applying only to those commodities the Fréchet derivatives of U with respect to which are not identically zero. Similar remarks apply to the case of production functions for firms.

Obviously, this definition only makes sense for $\theta \geq v$. We set $r(\theta,v,t) = 0$ for $v > \theta$, and take $r(\theta,v,t)$ for $v \leq t$ to be actual purchases made at time v which will also be denoted $\bar{r}(\theta,v)$. It is not assumed that planned and actual purchases coincide at $v = t$; this is what disequilibrium is all about.

As stated, the household assumes $p(\theta,t)$ to be independent of t.[1] It therefore faces the budget constraints:

$$(5.1) \qquad \int_v^\infty p(\theta,t)r(\theta,v,t)d\theta = 0 \qquad \forall \quad v \geq t \quad .^2$$

Although all trades must take place for money, this is not made explicit.

At time t, the household expects to have available at time θ a stock of then currently dated commodities. We denote this expected stock by $x(\theta,\theta,t)$, using the repeated first argument to remember that it is the stock of θ-dated commodities expected to be on hand at θ. The origins of this available stock will be considered in a moment.

At time θ, after trade (which we assume to take no time) the household will be able to do one of two things with its available stock. It can either consume it or it can store it, to the extent it is not instantaneously perishable. Accordingly, the household is constrained by:

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[1] This is one of the reasons why it is inconvenient to have future money explicitly bear a different price than does spot money.

[2] Because of the argument-heavy notation, I have not distinguished transposition of vectors in taking inner-products, since the meaning is clear.

$$x(\theta,\theta,t) = c(\theta,t) + y(\theta,t) \qquad \forall \ \theta \geq t \tag{5.2}$$

where $y(\theta,t)$ is the n-vector of the amounts of commodities which the household plans at t to put into storage as of θ. Naturally, only currently dated commodities can actually be stored.

The results of such storage activities are given by a vector of storage functionals, $\phi(y(v,t),\theta)$, where $v < \theta$, assumed Fréchet differentiable in $y(v,t)$ and continuous in θ. This gives the amounts of the various commodities which will be available to take out of storage at θ if the input plans for storage $y(v,t)$ are carried out. (For $v \leq t$, $y(v,t) = \bar{y}(v,t)$ denotes actual storage inputs, coinciding with planned inputs at $v = t$.)

It is convenient to incorporate in the definition of $\phi(y(v,t),\theta)$ rather more than just simple storing of goods, however. We have already done so in discussing bonds above. In addition, there is no reason to think of storage as simply placing a good in a warehouse and then taking it out again. Certain commodities, such as labor may be expended in so doing. These are to be thought of as placed in storage with no later output. Perishable commodities can be likewise considered.

Now, the stock of spot goods expected to be available at θ, $x(\theta,\theta,t)$, can come from four sources. First, some of that stock may already have been received from outside by t; we denote that stock by $\bar{x}(\theta,t)$ and consider it further below. Second, it may be acquired through purchase between t and θ. Third, it may be available from storage. For ordinary goods, this exhausts the possibilities; for money, however, it does not because the household may own shares in firms and expect to share in their profits.

Firms, discussed more fully in the next section, earn profits in money and pay dividends in money. Since present and future money have the same price, however, there is no reason why such profits and dividends cannot be paid in future as opposed to present money, although it does not matter. Hence, we define $s(\theta,t)$ as a vector which has identically zero functions for its first $n-1$ components and has for its nth component a function showing the household's share of the *total* profits earned by the firms in which it owns shares which it expects to receive in money dated θ. Similarly, we denote by $d(\theta,t)$ a vector which has identically zero functions for its first $n-1$ components and has for its nth component a function showing the total amount of those profits already paid to the household as of time t. Thus the last component of $s(\theta,t) - d(\theta,t)$ shows the household's share of *undistributed* profits which it expects to receive over time. There is no harm, however, in thinking of all this as involving functions which are non-zero only for $\theta = t$, that is, of having profits and dividends only involve current money, so long as one recalls the notational convention already adopted as to integrals involving money.

We can now write the available stock, $x(\theta,\theta,t)$, in terms of its sources as:

$$(5.3) \qquad x(\theta,\theta,t) = \bar{x}(\theta,t) + \int_t^\theta r(\theta,v,t)dv + \phi(y(v,t),\theta) + s(\theta,t) - d(\theta,t)$$

$$\forall \ \theta \geq t.$$

Combining this with (5.2) gives another set of constraints within which utility maximization takes place.

It remains to describe the origins of the actual acquired stock of goods dated θ on hand as of t, $\bar{x}(\theta,t)$. This consists of the household's actual stock of such good as of some arbitrary starting date, time 0 (this may be thought of as the household's original endowment) plus actual purchases and actual dividends.

$$(5.4) \qquad \bar{x}(\theta,t) = \bar{x}(\theta,0) + \int_0^t \bar{r}(\theta,v)dv + d(\theta,t) - d(\theta,0) \quad .$$

Here, $d(\theta,0)$ is subtracted since any dividends already paid by time zero are already counted in $\bar{x}(\theta,0)$. $\bar{x}(\theta,t)$ is assumed a continuous profile.

The relevant Lagrangian is given by:

$$(5.5) \qquad \Lambda_1 = U(c(\theta,t) - \int_t^\infty \lambda(v,t) \int_v^\infty p(\theta,t)r(\theta,v,t)d\theta dv$$

$$+ \int_t^\infty \mu(\theta,t) \{\bar{x}(\theta,t) + \int_t^\theta r(\theta,v,t)dv + \phi(y(v,t),\theta)$$

$$+ s(\theta,t) - d(\theta,t) - c(\theta,t) - y(\theta,t)\} \, d\theta \ ,$$

where $\lambda(v,t)$ and $\mu(\theta,t)$ are Lagrange multiplier functions (the latter being a vector).

We assume that the household's maximization problem has a solution, unique as regards $c(\theta,t)$ and $y(v,t)$ (but not regards $r(\theta,v,t)$), which occurs at a stationary point of (5.5), and which is continuous in θ (recall that $p(\theta,t)$ is so continuous). It is evident, either from elementary considerations or from examination of the constraints and the first-order conditions given below, that the optimal consumption profile, $c(\theta,t)$, and the optimal storage profile, $y(\theta,t)$, depend only on prices and on wealth, $w(t)$ defined as:

$$w(t) \equiv \int_t^\infty p(\theta,t)\,\{\bar{x}(\theta,t) + s(\theta,t) - d(\theta,t)\}\,d\theta \tag{5.6}$$

rather than on stocks and undistributed profits directly. We shall assume:

Assumption 5.1: The solution in $c(\theta,t)$ and $y(\theta,t)$ to the household's optimization problem is continuous in $w(t)$ and weak* continuous in $p(\theta,t)$.

Weak* continuity in the prices seems fairly natural. It essentially says that if a sequence of price profiles approaches a limit in the sense that for every consumption program the net cost approaches what it would be with the limiting price profile, then the optimal consumption and storage profiles must also approach the optimum under the limiting price profile. We shall need this property in the stability proof below.

Note that all this involves constraints on the storage functional $\phi(y(v,t),\theta)$; in particular, it rules out constant returns. This is necessary here as it will be for firms (and was in [6] and [7]), but is of course not so restrictive for households. It is required, among other reasons, if excess demands are to be functionals in the prices rather than transformations.

Denoting Fréchet derivatives by subscripts in the obvious way, the first-order conditions for a maximum are (ignoring corner solutions, for simplicity):

$$U_c(c(\theta,t)) = \mu(\theta,t) \qquad \theta \geq t\ , \tag{5.7}$$

$$\lambda(v,t)p(\theta,t) = \mu(\theta,t) \qquad \theta \geq v \geq t\ , \tag{5.8}$$

and

$$\int_v^\infty \mu(\theta,t)\phi_y(y(v,t),\theta)\,d\theta = \mu(v,t) \qquad v \geq t\ . \tag{5.9}$$

These have the following direct implications. First, it is clear from (5.8) that $\lambda(v,t)$ is in fact independent of v so that we may write:

$$(5.10) \qquad \lambda(v,t) = \lambda(t) \qquad v \geq t .$$

This simply reflects the fact that since the household expects prices to remain unchanged it does not care when before θ it makes its purchases of goods dated θ. This fact also means that $r(\theta,v,t)$ is not uniquely determined, a subject to which we shall have to return in considering the Present Action Postulate.

Next, combining (5.7) and (5.8), we obtain:

$$(5.11) \qquad U_c(c(\theta,t)) = \lambda(t)p(\theta,t) \qquad \theta \geq t ,$$

which is the expected result of marginal utilities proportional to prices.

Finally, combining (5.8) and (5.9) yields:

$$(5.12) \qquad \int_v^\infty p(\theta,t)\phi_y(y(v,t),\theta)\, d\theta = p(v,t) \qquad \theta \geq v \geq t ,$$

which can be interpreted as stating that marginal revenue product equals factor price in the storage activity.

6. Behavior of Firms

Firms will later be subscripted f, but the subscript will be omitted until needed much later on. So far as possible, I use a parallel notation for firms and for households for mnemonic convenience. The differences between them are indicated by the different subscripts when necessary later.

The firm at time t seeks to maximize its profits, denoted by $\pi(t)$ and given by:

$$\pi(t) = -\int_t^\infty \int_t^\theta p(\theta,t)r(\theta,v,t)dvd\theta + \bar{\pi}(t) \ , \tag{6.1}$$

where $\bar{\pi}(t)$ denotes profits already attained at time t (defined explicitly below) and, as with households, $r(\theta,v,t)$ denotes the purchases of goods dated θ which the firm plans at time t to make at time v.[1]

There is one difference in the notation for firms, however. Since firms keep score as to profits in money, $r(\theta,v,t)$ for a firm always has its nth component zero as does $\bar{r}(\theta,v)$ below.[2]

Actual profits, $\bar{\pi}(t)$ are given by:

$$\bar{\pi}(t) = -\int_0^\infty \int_0^t p(\theta,v)\bar{r}(\theta,v) \ dvd\theta + \bar{\pi}(0) \tag{6.2}$$

where $\bar{r}(\theta,v)$ denotes actual purchases of goods dated θ made at time v. (Naturally, $\bar{r}(\theta,v) = 0$ for $v > \theta$.)

In its profit maximization, the firm has access to a production functional:

$$b(\theta,t) = \phi(y(v,t),\theta) \tag{6.3}$$

where $b(\theta,t)$ denotes the output of goods expected available at θ as of time t and $y(v,t)$ denotes the profile of net inputs to be made at time v for

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[1] I apologize for departing from the usual convention of measuring outputs positively and inputs negatively, but the experience of [6] and [7] shows that to do otherwise leads to a bewildering array of minus signs.

[2] The firm might also make profits if money can be produced directly by it. It is notationally simpler to ignore this possibility and to assume that money does not enter the production process. Thus, despite the fact, for example, that output vectors are defined to have n components below, the nth component should be taken as 0. Similar remarks apply to other vectors.

$v \geq t$ or already made at v for $v \leq t$ (these are assumed to coincide at $v = t$).[1] $\phi(y(v,t),\theta)$ is assumed Fréchet differentiable with respect to $y(v,t)$ and continuous in θ.

Some discussion of this is in order. First, I have deliberately adopted the same notation already used for the storage activities of households because there is no essential difference between the two things. Indeed, it would be possible to treat each household as wholly owning a firm whose production functional describes the household's storage technology. The first-order conditions for the household imply (5.12) which implies that it treats its storage activity as would a profit maximizing firm.

Second, it would of course be more natural to have the output of a particular commodity depend on the inputs used in the production of that commodity rather than on total inputs (This is true for household storage as well.). To do this explicitly would needlessly burden the notation and the appearance of both t and θ in the production functional can be taken to indicate that the firm optimally allocates its inputs, given the price profile as of t, making outputs implicitly depend only on total inputs.

The firm is constrained in its maximization process by:

$$(6.4) \qquad \bar{x}(\theta,t) + \int_t^\theta r(\theta,v,t)\,dv + \phi(y(v,t),\theta) - y(\theta,t) = 0$$

$$\forall \;\; \theta \geq v \geq t \;,$$

where $\bar{x}(\theta,t)$ denotes the actual stock of goods dated θ already purchased as of t and is given by:

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[1] Net inputs are involved here as for the household because of the possibility that the firm may withdraw inputs from the production process.

$$(6.5) \qquad \bar{x}(\theta,t) = \bar{x}(\theta,0) + \int_0^t \bar{r}(\theta,v)\,dv \quad .$$

$\bar{x}(\theta,t)$ is assumed continuous in θ.

The relevant Lagrangian for the firm is given by:

$$(6.6) \qquad \Lambda_2 = -\int_t^\infty \int_t^\theta p(\theta,t)r(\theta,v,t)\,dv d\theta + \bar{\pi}(t)$$

$$+ \int_t^\infty \gamma(\theta,t)\,\{\bar{x}(\theta,t) + \int_t^\theta r(\theta,v,t)\,dv + \phi(y(v,t),\theta) - y(\theta,t)\}\,d\theta$$

where $\gamma(\theta,t)$ is a vector of Lagrange multiplier functions.

We assume that the firm's maximization problem has a solution, unique as regards $y(v,t)$, which occurs at a stationary point of (6.6) and which is continuous in θ. It is evident, either from elementary considerations or from examinations of the constraints and the first-order conditions below, that the solution in $y(v,t)$ depends only on the prices and on the value of the firm's holdings, denoted by $w(t)$ and given by:

$$(6.7) \qquad w(t) \equiv \int_t^\infty p(\theta,t)\bar{x}(\theta,t)\,d\theta$$

rather than on $\bar{x}(\theta,t)$ directly. We further assume:

Assumption 6.1: The solution in $y(\theta,t)$ to the firm's optimization problem is continuous in $w(t)$ and weak* continuous in $p(\theta,t)$.

As before, this unfortunately rules out constant returns. Handling that leading case appears to require quite different techniques and it remains to be worked out.

The first-order conditions for a maximum are (ignoring corner solutions for simplicity):

$$p(\theta,t) = \gamma(\theta,t) \qquad \theta \geq t \tag{6.8}$$

and

$$\int_v^\infty \gamma(\theta,t)\phi_y(y(v,t),\theta)\, d\theta = \gamma(v,t) \qquad \theta \geq v \geq t\ . \tag{6.9}$$

Combining these yields:

$$\int_t^\infty p(\theta,t)\phi_y(y(v,t),\theta)\, d\theta = p(v,t) \qquad v \geq t\ , \tag{6.10}$$

which parallels (5.12) and has the interpretation that marginal revenue product must equal factor price.

Given its actual profits, $\bar{\pi}(t)$, the firm pays out dividends to its owners. In addition, of course, those owners expect to share in the expected profits of the firm not yet earned, that is in $\pi(t) - \bar{\pi}(t)$. As described in the previous section, both announced dividends and expected dividends are most conveniently thought of as paid out over time. Thus, we denote by $\bar{g}(\theta,t)$ an n-component vector whose first n-1 components are identically zero and whose last component gives the schedule of dividends already announced at time t paid out or to be paid out over different times. Similarly, $g(\theta,t)$ denotes an n-component vector differing from $\bar{g}(\theta,t)$ only in that the last component gives the schedule of total dividend payments expected to be made by the firm (announced and unannounced). It will be convenient, though not really essential, to suppose that all shareowners have the same expectations as to the schedule of such payments.

We shall assume that both $g(\theta,t)$ and $\bar{g}(\theta,t)$ are continuous in $\pi(t)$, $\bar{\pi}(t)$ and $w(t)$ and weak* continuous in $p(\theta,t)$. It is further natural to assume that the firm cannot pay out more than it has so that:

$$(6.11) \qquad \bar{G}(t) \equiv \int_{-\infty}^{\infty} p(\theta,t)\bar{g}(\theta,t)\,d\theta \leq \bar{\pi}(t) \quad ; \quad \int_{-\infty}^{\infty} p(\theta,t)g(\theta,t)\,d\theta = \pi(t) \ .$$

(Of course, $g(\theta,t) = \bar{g}(\theta,t)$ for $\theta < t$.) We shall later show that all profits are distributed in equilibrium, but there is no harm in allowing them to be retained for liquidity purposes during the adjustment process (Indeed, to do otherwise might require capital levies on a firm's stockholders.).

We can now connect this to the theory of the household by assuming that the hth household owns a fixed share, k_{hf}, of the fth firm, where $k_{hf} \geq 0$ and $\sum_h k_{hf} = 1$ (Since everyone has the same expectations about profit opportunities, there is little point in having a market in shares.). Then, introducing the subscripts in the obvious way:

$$(6.12) \qquad s_h(\theta,t) = \sum_f k_{hf}g_f(\theta,t) \quad ; \quad d_h(\theta,t) = \sum_f k_{hf}\bar{g}_f(\theta,t) \ .$$

7. Closure Equations and Walras' Law

Since actual sales must be matched by actual purchases for ordinary goods and since money is not produced, we have the closure equations:

$$(7.1) \qquad \sum_h \bar{r}_{hi}(\theta,t) + \sum_f \bar{r}_{fi}(\theta,t) = 0 \qquad (i=1,\dots,n-1)$$

$$\sum_h \int_t^\infty \bar{r}_{hn}(\theta,t)\,d\theta + \sum_f \dot{\bar{\pi}}_f(t) = 0 \quad .$$

Further, combining the household's constraints (5.2) and (5.3), premultiplying by $p(\theta,t)$, integrating over θ, summing over households, using the budget constraint (5.1), the distributive equations (6.12), the definition of profits (6.1), and the constraint (6.4), we obtain Walras' Law for this economy, namely:

$$(7.2) \qquad \int_t^\infty p(\theta,t)\,\{\sum_h (c_h(\theta,t) + y_h(\theta,t) - \bar{x}_h(\theta,t) - \phi^h(y_h(v,t),\theta))$$

$$+ \sum_f (y_f(\theta,t) - \bar{x}_f(\theta,t) - \phi^f(y_f(v,t),\theta))\}\,d\theta$$

$$+ \sum_f (\bar{G}_f(t) - \bar{\pi}_f(t)) = 0 \ .$$

Note that the firm's excess demand for money includes its desire to hold retained earnings.

8. The Present Action Postulate

We have already had occasion to remark that participants in this model do not have a uniquely determined schedule of intended purchases, $r(\theta,v,t)$. The reason for this is that each participant expects prices to remain as they are now; hence, while the *total* of his future purchases of each commodity is determined by the solution to his optimization problem, the *timing* of those purchases is not. If this is not at once obvious, one need only observe that, given any optimal purchase profile, $r(\theta,v,t)$, one can obtain another from it by planning to purchase more of a particular future commodity than one needs and to resell the excess at the purchase price at some future time (before the date on the commodity comes due). In other words, the middle argument in $r(\theta,v,t)$ is largely inessential.

Up to now, this has presented no particular problem. When we come to a consideration of the adjustment processes involved in the model, however, such indeterminacy cannot be allowed to persist; indeed, its resolution is a matter of central importance. This is because it is natural to take prices as driven by excess demands (either through some wholly mysterious process or through the conscious actions of individual dealers). There is no sense, however, in which prices can change in response to excess demands which remain merely gleams in the eye of the demander. It is the attempt to exercise excess demand which affects price; hence we cannot hope to have an adjustment model which lacks some specification of the way in which planned purchases are reflected in present actions.[1]

For our purposes, it is not necessary to be precise as to how current purchases are determined from future plans; it is merely necessary to assume that they are so determined in a continuous manner and that non-zero future needs result in some immediate attempt to satisfy them. Thus, denoting the ith component of $r(\theta,v,t)$ by a subscript, we assume:

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[1] The recognition of the fact that planned purchases must be translated into actual attempts to buy to affect prices also lies at the heart of the institutional assumptions concerning money and no bankruptcy introduced by Arrow and Hahn [1, Ch. XIII] into these models and continued here. For a more elaborate discussion, see Fisher [8].

Assumption 8.1 (Present Action Postulate): For each participant (household or firm), there exists a set of n-1 continuous and sign-preserving functions, $F^i(.)$, $i = 1, \ldots, n-1$, which do not approach zero except as their arguments do, such that:

$$(8.1) \qquad r_i(\theta,t,t) = F^i\left(\int_t^{\theta} r_i(\theta,v,t)\,dv\right) .$$[1]

Note that we shall interpret $r(\theta,t,t)$ as post-trade purchase demand, so that it is an unsatisfied demand. This means that we have assumed that each participant immediately acts so as to satisfy his future needs and does so (with trade taking place instantaneously) until he has either satisfied those needs or is unable to trade further.

What can be said about this assumption other than that it is implicit in all existing stability models? I have elsewhere, [7] and [8], discussed it extensively and so shall be somewhat briefer here.

In the first place, we have already remarked that the presence of dated commodities makes the Present Action Postulate more palatable than it would otherwise be. This is so even for households where we require that a household expecting to require toothpaste in 1984 now attempt to purchase 1984 toothpaste rather than now attempting to acquire spot toothpaste. It is much more important for firms, however, for it prevents us from having to assume that firms now begin to sell outputs on the spot market which will not be produced for years; further, we can allow a firm to acquire a given commodity for use as an input even though it expects to be a net supplier of that commodity (with a later date) later on. Indeed, the introduction of dating makes the problems associated with the Present Action Postulate essentially no greater for firms than for households.

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[1] Remark: It is not necessary that present purchase attempts depend only on the integral of future desired purchases, so long as the properties of $F^i(.)$ as a function of that integral are retained; (7.1) gives the simplest version.

Nevertheless, those problems are not negligible. Basically, firms and households are indifferent as to the timing of their purchases; we are insisting that they choose a particular kind of action from a set of equally optimal actions. Why should they do this?

The answer inevitably lies somewhat outside the models of the household and the firm so far given. Those models are essentially equilibrium models and the kinds of justification for the Present Action Postulate that come readily to mind require some consciousness of disequilibrium. Thus, for example, present action may be optimal if one realizes that one may not in fact be able to complete ones purchases. More important than this, perhaps, is the fact that experience with attempts to buy just a little will show, given the Hahn Process Assumption (described formally below), that prices are moving perversely so that hanging back from the market is unlikely to be seen as a successful way to speculate. Both these reasons will be reinforced by the realization that trading in goods dated θ must be completed by θ.

Nevertheless, these arguments, however persuasive, import into the motivations of the participants some consciousness of disequilibrium and of uncertainty which is conspicuously absent in their price expectations and in the derivation of their optimal strategies.

9. The Hahn Process and Trading Rules

As already discussed, the basic trading rule we shall use is that of the Hahn Process, which is extremely natural when markets are distinguished by dealers. The basic assumption is that there are not at the same (post-instantaneous trade) time both unsatisfied suppliers and unsatisfied demanders of the same commodity. Formally, define:

$$R(\theta,t) = \sum_h r_h(\theta,t,t) + \sum_f r_f(\theta,t,t) , \quad \theta \geq t . \tag{9.1}$$

We assume:

Assumption 9.1 (Hahn Process):

a) For all households, h, and ordinary commodities, i=1,...,n-1, $r_{hi}(\theta,t,t)R_i(\theta,t) \geq 0$, and $r_{hi}(\theta,t,t)R_i(\theta,t) > 0$ unless $r_{hi}(\theta,t,t) = 0$.

b) For all firms, f, and ordinary commodities, i=1,...,n-1, $r_{fi}(\theta,t,t)R_i(\theta,t) \geq 0$, and $r_{fi}(\theta,t,t)R_i(\theta,t) > 0$ unless $r_{fi}(\theta,t,t) = 0$.

As discussed earlier, this really makes most sense in a framework in which all trades take place for money and the assumption is taken to apply to "active" excess demands, that is to all offers to sell and to all offers to buy which are backed up by money. It is then necessary to assume in order to make the rest of the analysis go through that nobody ever runs out of money. We are implicitly adopting all of this.

It is not necessary to be explicit about the way in which actual trades, $\bar{r}(\theta,t)$, take place. We need only assume that they do so in a way consistent with the Hahn Process and satisfying appropriate Lipschitz conditions. One way to ensure this is to have participants on the short side of any market ration their transactions over participants on the long side in proportion to the demands of the latter (this is done explicitly in Arrow and Hahn [1, Ch. XIII]), but this is not the only way.

One other matter needs to be discussed before proceeding, namely, the assumption already made in passing that households and firms are always able to fulfill their consumption, storage, and production plans as regards currently dated commodities. (We assumed this in assuming that actual and planned consumption, storage, and production profiles coincided at t.) This assumption seems necessary in order to avoid awkward discontinuities caused by the sudden realization that plans cannot be fulfilled. There are (at least) two ways in which it can be violated.

In the first place, since participants expect their transactions to be completed, it might be the case, for example, that a household holds no stock of a particular dated commodity but plans to have its consumption come entirely from current purchases. Since it may not be able to make those purchases, it may find itself suddenly unable to fulfill its consumption plan (The discontinuities involved here are very similar to those created by dated commodities if continuity of prices is not assured.).

Second, even if a participant does hold buffer stocks with which it expects to fulfill its plans, it might be the case that the stock of one such commodity so held consists of commitments by firms to produce and deliver that commodity by the time its date comes due. Out of equilibrium, however, it is possible that firms may not be able to satisfy those commitments since they may not be able to acquire the necessary inputs.

Obviously, in this latter situation, one might expect the prices for the commodities in question to rise very quickly as short firms attempt to cover their commitments. While it is possible to build such behavior into the price adjustment rules given in the next section (by letting the rate of change of price depend on $\theta - t$), it does not appear particularly simple to ensure that this always results in short positions being covered by the time delivery is due.

Such price behavior, moreover, makes sense only if we suppose that individuals recognize the possibility of not being able to complete their desired transactions. If we allow that, however, we might as well also allow firms to hold stocks of outputs against the possibility that they will be caught short; similarly, we might as well allow households and firms to hold buffer stocks of inputs into consumption, storage, or production. We could, indeed, introduce constraints into the maximization problems requiring stocks to remain greater than zero for certain goods without altering the first-order conditions (other than the constraints) or the rest of the analysis. For some goods, at least, this might solve the problem.[1] It would be artificial, however. On the whole, it seems better to beg these questions entirely and assume that the problems involved do not arise. Further work here seems very desirable.

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[1] It would not solve it for all because some commodities are instantaneously perishable (services). Even for commodities which can be stored the problem is not as simple as it may appear. Even though inputs are flows and thus cannot exhaust stocks, the "stock" of a particular dated commodity is a flow when commodities are continuously dated in the same sense that inputs are.

10. Price Adjustment

We adapt the conventional price adjustment assumptions to the present model. Here, and later, dots denote derivatives with respect to t.

Assumption 10.1 (Price Adjustment): For each of the non-monetary commodities, $i=1,\ldots,n-1$, there exists a continuous and sign-preserving function, $H^i(.)$, which does not approach zero except as its argument does, such that for all $\theta \geq t$, $\dot{p}_i(\theta,t) = H^i(R_i(\theta,t))$ unless $p_i(\theta,t) = 0$ and $R_i(\theta,t) < 0$ in which case $\dot{p}_i(\theta,t) = 0$.

Note the feature that it is the same function, $H^i(.)$, which applies for all values of θ. In fact, this can be generalized to allow continuous dependence on θ. The crucial feature is continuity in θ which allows continuous price profiles to remain continuous. It is hard to see why this should be the case unless the same individual controls price adjustment for a given commodity for all θ. This is, of course, the dated commodities problem with which we began.

Recall that $p(\theta,t)$ becomes fixed as t passes θ.

We assume that the adjustment mechanisms satisfy the appropriate Lipschitz conditions so that there exist solutions to the partial differential equations involved which are continuous in the initial conditions. This assumption is also made as regards the other fundamental variables in the system, acquired stocks, $\bar{x}(\theta,t)$.[1]

---

[1] Henry [11] and [12] has shown that the restriction of prices to the non-negative orthant presents no essential problem here, at least for the usual case of a finite number of commodities. (See also Drèze [2].) The present case can be regarded as having a continuum of commodities or as involving partial differential equations. It seems clear that Henry's work can be extended to cover this.

11. Decline of Target Profits

We now return to the theory of the firm and show that target or expected profits, $\pi(t)$, decline unless the firm is able to complete all its transactions (with the exception of disposing of free goods). The basic reason for this is the Hahn Process assumption which makes goods demanded go up and goods supplied go down in price,[1] together with the fact that the irreversibility of production commitments cannot make things better.

The proof is correspondingly in two steps. First, observe that the firm's maximized profits at time t depend on those things which the firm takes as given. These are, prices, $p(\theta,t)$, profits already earned, $\bar{\pi}(t)$, and the value of the actual stock of already purchased goods, $\bar{x}(\theta,t)$. In addition, the firm is constrained by its own past input and output decisions, that is, by the part of the profile $y(v,t)$ already committed and its consequent effects on $\phi(y(v,t),\theta)$. When the firm decides on its optimal program at time t, it takes actions which further constrain its future behavior by choosing $y(t,t)$. We first consider what we shall term "replannable" profits, denoted by $\pi^*(t)$ which differ from target profits, $\pi(t)$, in that replannable profits are the profits which could be earned if the firm could continually go back to t and remake its production decision from then on so as to reoptimize its behavior given new prices and new acquired stocks.

---

[1] That this implies the decline of target profits as well as of target utilities was first observed in [6].

We shall refer to a position in which the firm completes all its transactions with the possible exception of disposing of free goods as an equilibrium for the firm.

Lemma 11.1: For every firm, $\dot{\pi}^*(t) \leq 0$ and $\dot{\pi}^*(t) < 0$ unless the firm is in an equilibrium for it.

Proof: By the Envelope Theorem for constrained maximization, we can evaluate $\dot{\pi}^*(t)$ by differentiating the Lagrangian (6.6) with respect to the things not under the firm's control.[1]

$$(11.1) \qquad \dot{\pi}^*(t) = \dot{\Lambda}_2 = -\int_t^{\infty}\int_t^{\theta} \dot{p}(\theta,t)r(\theta,v,t)\,dv\,d\theta + \dot{\bar{\pi}}(t) + \int_t^{\infty} \gamma(\theta,t)\bar{r}(\theta,t)\,d\theta \quad ,$$

recalling that we are assuming that changes in the production plan of the firm can be readjusted so as to be under the firm's control.[2]

Differentiating the definition of realized profits, $\bar{\pi}(t)$, given in (6.2) and using the first-order conditions (6.8) we find:

---

[1] See Fisher [7] for a brief discussion of the extension of the Envelope Theorem to problems involving functionals.

[2] It might be supposed that there would be further terms in (11.1) stemming from the derivatives of the integrals in Λ_2 with respect to the lower limits of integration. In fact, those terms do not appear because all the integrals in (6.6) are items under the firm's control. This can be verified, if required, by evaluating those terms in the light of the constraint (6.4) and the first-order conditions (6.8).

$$(11.2) \qquad \dot{\bar{\pi}}(t) = - \int_0^\infty p(\theta,t)\bar{r}(\theta,t)\, d\theta = - \int_0^\infty \gamma(\theta,t)\bar{r}(\theta,t)\, d\theta \quad ,$$

so that

$$(11.3) \qquad \dot{\pi}^*(t) = - \int_t^\infty \int_t^\theta \dot{p}(\theta,t)r(\theta,v,t)\, dv d\theta$$

$$= - \int_t^\infty \dot{p}(\theta,t) \left\{ \int_t^\theta r(\theta,v,t)\, dv \right\} d\theta \quad .$$

By the Present Action Postulate (Assumption 8.1), however, the component of the inner integral on the far right-hand side of (11.3) corresponding to the ith commodity ($i=1,\ldots,n-1$) has the same sign as $r_i(\theta,t,t)$. Further, by the Hahn Process Assumption (Assumption 9.1), if $r_i(\theta,t,t)$ is not zero, it has the same sign as $R_i(\theta,t)$. Finally, by the Price Adjustment Assumption (Assumption 10.1), that is also the sign of $\dot{p}_i(\theta,t)$, unless $p_i(\theta,t) = 0$ and $r_i(\theta,t,t) < 0$. The lemma is now immediate.

It is now easy to prove:

Theorem 11.1: For every firm, actual target profits, $\pi(t)$, are non-increasing through time and are strictly declining unless the firm is at an equilibrium for it.

Proof: Consider a small positive time interval, dt. At t, $\pi(t) = \pi^*(t)$, but $\pi(t + dt) \leq \pi^*(t + dt)$, since $\pi(t + dt)$ is the maximum profit which can actually be made at $(t + dt)$ and $\pi^*(t + dt)$ is the maximum profit which could be made at $(t + dt)$ if the firm could go back and redo its production decisions during $[t, t + dt]$. By Lemma 11.1, however, for small enough dt, $\pi^*(t + dt) \leq \pi^*(t)$ with the strict inequality holding unless the position at t is one of equilibrium for the firm. This proves the theorem.[1]

We shall assume $\pi(t)$ bounded below.

12. Decline of Target Utilities

With this result in hand, we are almost ready to prove the parallel result for target utilities. Before doing so, however, we need to add one more assumption.

We have already assumed in (5.1) that the household believes it can acquire goods by trade at constant prices only by giving up goods of equal value. We now assume that it is correct in this:

Assumption 12.1 (No Swindling):
$$\int_t^\infty p(\theta,t)\bar{r}(\theta,t)\,d\theta = 0 .$$

As before, the analysis is divided into two parts. We first consider replannable utility, U^*, the utility which would be achieved if the household could constantly revise its consumption and storage plans from t onward. U^* depends on prices, stocks received from outside and undistributed profits.

---

[1] The reason that dt must be taken small is that π^* is defined relative to t and Lemma 11.1 holds only at t. This is because the excess demands which would arise from optimizing π^* will differ, in general, after t from those which arise in optimizing π itself, and it is the latter which help drive the prices.

We shall refer to a position in which the household completes all its transactions (with the possible exception of disposing of free goods) as an equilibrium for the household.[1]

Lemma 12.1: For every household, $\dot{U}^* \leq 0$, and $\dot{U}^* < 0$ unless the household and every firm in which it owns a positive share are in equilibria for them.

Proof: By the Envelope Theorem, we can evaluate $\dot{U}^*$ by differentiating the Lagrangian (5.5) with respect to the things outside the control of the household. Thus:

$$\dot{U}^* = \dot{\Lambda}_1 = -\int_t^\infty \lambda(v,t) \int_v^\infty \dot{p}(\theta,t) r(\theta,v,t)\, d\theta dv \tag{12.1}$$

$$+ \int_t^\infty \mu(\theta,t) \{\bar{r}(\theta,t) + \dot{d}(\theta,t) + \dot{s}(\theta,t) - \dot{d}(\theta,t)\}\, d\theta$$

recalling that consumption and storage can be readjusted from t onwards.[2]

---

[1] Note that an equilibrium for the household cannot occur unless all profits to which it is entitled are distributed. This is because the household behaves as though it has the undistributed profits to spend. Hence it will either be purchasing with them in which case there is an unsatisfied transaction or it will wish to consume or store them. But we have assumed that current consumption and storage plans are satisfied. In either case, profits must be distributed. This will show up again when we define equilibrium and consider the implications of Walras' Law (7.2).

[2] A comment similar to that made in the case of firms applies to terms stemming from differentiation with respect to limits of integration.

Using the first-order conditions, (5.6) - (5.10), this becomes:

$$(12.2) \qquad \dot{U}^* = -\lambda(t)\int_t^\infty \dot{p}(\theta,t)\left\{\int_t^\infty r(\theta,v,t)\,dv\right\}d\theta$$

$$+\lambda(t)\int_t^\infty p(\theta,t)\dot{s}(\theta,t)\,d\theta + \lambda(t)\int_t^\infty p(\theta,t)\bar{r}(\theta,t)\,d\theta .$$

The third term in this is zero by the No Swindling Assumption, however, while the integral in the second term is the rate of change of the household's share of profits of the firms it owns (recall that $s(\theta,t)$ has only its nth component nonzero and that $p_n(\theta,t) = 1$). By Theorem 11.1, this is nonpositive and is strictly negative unless all those firms are in equilibria for them. Since $\lambda(t)$ is obviously positive, it remains only to show that the double integral multiplied by $-\lambda(t)$ in the first term is nonnegative and is strictly positive unless the household is at an equilibrium for it.

This is readily done, however. By the Present Action Postulate (Assumption 8.1), the component of the inner integral corresponding to the ith commodity ($i=1,\ldots,n-1$) has the same sign as $r_i(\theta,t,t)$. Further, by the Hahn Process Assumption (Assumption 9.1), if $r_i(\theta,t,t)$ is not zero, it has the same sign as $R_i(\theta,t)$. Finally, by the Price Adjustment Assumption (Assumption 10.1), that is also the sign of $\dot{p}_i(\theta,t)$, unless $p_i(\theta,t) = 0$ and $r_i(\theta,t,t) < 0$. The desired result now follows immediately.

An argument identical to that used to prove Theorem 11.1 from Lemma 11.1 now shows:

Theorem 12.1: For every household, actual target utility, U(.) is nonincreasing through time and is strictly decreasing unless the household and all the firms in which it owns a positive share are in equilibria for them. We shall assume U(.) bounded below.

13. Properties of Equilibrium

The fundamental variables of the system are prices, $p(\theta,t)$, and acquired stocks, $\bar{x}(\theta,t)$. It will be convenient to adopt a notation for the vector of the latter listing $\bar{x}(\theta,t)$ for all firms and households (in some definite but not important order). We denote it by $\bar{X}(\theta,t)$.

When discussing equilibria, it will be convenient to suppress the time argument and speak of $p(\theta)$ and $\bar{X}(\theta)$.

Now, since consumption, storage, and production take place out of equilibrium, the maximization problem facing each participant changes; that is, past consumption fixes some of the argument in the utility functional to be maximized, while past production and storage decisions affect the constraints within which present optimization takes place. It follows that the set of $(p(\theta), \bar{X}(\theta))$ which are equilibria will change over time depending on the production and consumption histories which occur. We thus define:

Definition 13.1: $(p(\theta), \bar{X}(\theta))$ is an equilibrium for t if and only if, given the history of the economy at t, with $p(\theta,t) = p(\theta)$ and $\bar{X}(\theta,t) = \bar{X}(\theta)$, for every household and firm:

$$(13.1) \qquad c(\theta,t) + y(\theta,t) - \bar{x}(\theta,t) - \phi(y(v,t),\theta) \leqq 0. \quad ^{1}$$

---

[1] I adopt the usual convention for vector inequalities in which $x > y$ means $x_i > y_i$, $x \geq y$ means $x_i \geq y_i$ but $x \neq y$, and $x \geqq y$ means $x_i \geqq y_i$ with $x = y$ permitted.

(The same expression holds for households and firms if we set consumption of firms at zero.)

It is easy to prove:

Lemma 13.1: If an equilibrium for t actually occurs at t, then:

a) For all participants and all nonmonetary commodities $i=1,\ldots,n-1$, $r_i(\theta,t,t) \leq 0$;[1] the strict inequality holds if and only if the corresponding inequality in (13.1) is also strict.

b) For all commodities, $i=1,\ldots,n$, a strict inequality in (13.1) implies $p_i(\theta,t) = 0$. Hence a strict inequality never occurs in the nth component of (13.1).

c) For every firm, $\bar{G}(t) = \bar{\pi}(t) = \pi(t)$.

Proof: Proposition (a) follows immediately for firms from the constraint (6.4) and the Present Action Postulate. For households, it follows from the constraints (5.2) and (5.3), the Present Action Postulate and the fact that undistributed profits are nonzero only for money. In other words, (a) simply says that excess demand for ordinary commodities can only be satisfied through purchases.

Proposition (b) and the first equality in (c) follow from examining Walras' Law (7.2) and observing that, in view of (13.1) and (6.11), every term in every summation in (7.2) is nonpositive. Hence every term must be zero. The second equality in (c) is evident.

Note that an equilibrium for t is an equilibrium for every participant. It is important to note that an equilibrium in this model (as in models without production and consumption) is an exhaustion of trading opportunities.

---

[1] Note that, in view of the Present Action Postulate (Assumption 8.1), this can be stated equivalently as
$$\int_t^\infty r_i(\theta,v,t)\, dv \leq 0 .$$

All trades are consummated and economic activity thenceforward consists of the carrying out of previously made plans and commitments. It would be interesting to have a stability model in which equilibrium involves continued trading.

On the other hand, it is of course true that there is no guarantee in the convergence proof below that equilibrium is reached in finite time, so that trading will in fact continue indefinitely (although it will disappear asymptotically).

This fact makes it necessary to consider the nature of equilibrium a bit more fully. If an equilibrium for t is actually attained at t, then it will remain an equilibrium for all later time. On the other hand, if an equilibrium for t is not actually attained at t, then the set of equilibria will change. Since convergence can only be established asymptotically, this means that we must think in terms of an asymptotic equilibrium set. This can most easily be done by recognizing that the left-hand side of (13.1) depends on prices, $p(\theta)$, acquired stocks, $\bar{X}(\theta)$, and actual past history. Denote that left-hand side by $z(p(\theta), \bar{X}(\theta), t)$ where the time argument stands for dependence on past history. We can now define:

Definition 13.2: $(p(\theta), \bar{X}(\theta))$ is an asymptotic equilibrium if and only if for every participant, all θ, and all commodities, $i=1,\ldots,n$,

$$(13.2) \qquad \limsup_{t\to\infty} \; z_i(p(\theta),\bar{X}(\theta),t) \leq 0 \quad .$$

We add some remarks. First, it is evident that the results of Lemma 13.1 apply asymptotically. Hence, in particular, for all goods for which

$p_i(\theta) \neq 0$, the limit superior in (13.2) can be replaced by an ordinary limit which will in fact be zero. The more complicated statement in (13.2) is due to the fact that the holdings of excess supplies of free goods need not approach a limit.

Second, we shall establish appropriate compactness below and prove that every limit point of the path of prices and acquired stocks is an asymptotic equilibrium. It follows that there is at least one asymptotic equilibrium. This may or may not be an equilibrium for some finite t which is actually attained at that time.

14. Boundedness

I have already discussed the issues surrounding the boundedness of prices in an earlier section. I shall now proceed to assume it directly.

Assumption 14.1: $p(\theta,t)$ remains in a bounded set.

The other fundamental variables of the system are the acquired stocks $\bar{X}(\theta,t)$. It turns out not to be necessary directly to assume these bounded, however, instead, it suffices to assume:

Assumption 14.2: For each participant the value of possessions, $w(t)$, remains bounded.

Given the boundedness of prices, this is weaker than assuming stocks themselves bounded. I have elsewhere, [6], [7], and [8], discussed the arguments for the stronger assumption.

15. Compactness and Quasi-Stability

We are going to use the decline of target utilities to take the sum of such utilities as a Lyapounov functional and establish quasi-stability. In other words, we shall show that every limit point of the path of prices and acquired stocks is an asymptotic equilibrium. This will do little good, however, unless we can show that such a limit point exists and, as it happens, the easiest way to show that appears to be tied up with the proof of quasi-stability itself. Hence we discuss both subjects together.

We begin with prices. By Assumption 14.3, $p(\theta,t)$ remains in a bounded set and we may obviously assume that it remains in the weak* closure of that set. By the Alaoglu Theorem, it follows that:

Lemma 15.1: The set of prices is weak* compact.

In what follows, any reference to limit points or to compactness is to be understood as being in the weak* topology as involves prices.

Similarly, since the value of possessions, $w(t)$, is a scalar, since we have assumed it to lie in a bounded set in Assumption 14.4, and since we can take it to lie in the closure of that set, it follows (denoting the vector of the $w(t)$ for all participants by $W(t)$):

Lemma 15.2: The set of values of possessions, $W(t)$, is compact.

The problem with proceeding directly, however, is that it is not enough to show that the time path of $W(t)$ has a limit point; we must show this (or almost this) for the time path of acquired stocks themselves, $\bar{X}(\theta,t)$ and here closure and boundedness are not enough. In fact, the time path of $\bar{X}(\theta,t)$ need not have a limit point, but the exception is inessential.

The exception concerns the treatment of free goods. As prices approach a weak* limit point and W(t) approaches a limit point, there is nothing to determine who holds the excess supplies of the goods which are free at the limiting prices. (We have already met this problem in defining an asymptotic equilibrium, Definition 13.2.) Clearly, however, this is inessential, since the holdings of excess free goods do not affect any consumption, storage, or production decisions. Hence, it is merely necessary to devise a way of conveniently dealing with such holdings.

This we do by defining what we shall call "essential stocks."

Definition 15.1: Given a price profile, $p(\theta)$, the vector of essential stocks relative to $p(\theta)$ for each participant, household or firm, is denoted by $q(\theta,t)$ and defined as follows. For every t, θ, and $i=1,\ldots,n$,

$$(15.1) \qquad q_i(\theta,t) = \bar{x}_i(\theta,t) \quad \text{if } p_i(\theta,t) \neq 0;$$

$$q_i(\theta,t) = \text{Min}\,\{\bar{x}_i(\theta,t)\ ,\ c_i(\theta,t) + y_i(\theta,t) - \phi_i(y(v,t),\theta)\}$$

$$\text{if } p_i(\theta,t) = 0\ .$$

(Recall that $c(\theta,t) = 0$ for firms.)

We denote the vector which lists essential stocks for all participants by $Q(\theta,t)$ and, when speaking of equilibria, by $Q(\theta)$. We shall speak of asymptotic equilibria as involving points in the space of $(p(\theta), Q(\theta))$ as:

Definition 15.2: $(p(\theta), Q(\theta))$ is an asymptotic equilibrium if and only if $Q(\theta)$ is defined relative to $p(\theta)$ and $(p(\theta), \bar{X}(\theta))$ would be an asymptotic equilibrium if $\bar{X}(\theta) = Q(\theta)$.

In a way, what all this amounts to is assuming free disposal of goods which are free at equilibrium.

We can now prove:

Lemma 15.3:

a) The time path of $(p(\theta,t), W(\theta,t))$ has a limit point (the limit being weak* in the case of $p(\theta,t)$).

b) To every limit point of the time path of $(p(\theta,t), W(\theta,t))$, say $(p(\theta), W(\theta))$, there corresponds a limit point to the time path of $Q(\theta,t)$, say $Q(\theta)$, where $Q(\theta,t)$ is defined relative to the limiting prices, $p(\theta)$.

c) Every such limit point, $(p(\theta), Q(\theta))$ is an asymptotic equilibrium.

Proof: (a) follows immediately from Lemmas 15.1 and 15.2. Now, $c(\theta,t)$ and $y(\theta,t)$ are continuous in $w(t)$ and weak* continuous in $p(\theta,t)$. It follows that, as $(p(\theta,t), W(\theta,t))$ approach $(p(\theta), W(\theta))$ along a convergent subsequence, $c(\theta,t)$ and $y(\theta,t)$ approach limits for every participant, say $c(\theta)$ and $y(\theta)$. Moreover, utilities are bounded below and nonincreasing. Hence utility for each participant approaches a limit and, by Theorem 12.1, this cannot occur unless for every agent the limit superior of excess demands for each commodity is nonpositive and the limit of excess demand strictly zero for all commodities with a positive price at the limiting $p(\theta)$. Thus, for all commodities for which the limiting price is positive,

$$(15.2) \qquad \text{Lim}\ \bar{x}_i(\theta,t) = \text{Lim}\ \{c_i(\theta,t) + y_i(\theta,t) - \phi_i(y(v,t),\theta)\}$$

and, since the right-hand side of (15.2) exists, $Q(\theta,t)$ must have a limit point. Evidently, from what has been said, using the sum of target utilities as a Lyapounov functional, every such limit point $(p(\theta), Q(\theta))$ must be an equilibrium, and the lemma is proved.

16. Global Stability

We are now nearly in position to prove the principal result of this paper, the global stability of the model. To be able to do this as regards prices, however, requires the following assumption:

Assumption 16.1 (Indecomposability): Let $S(p)$ be the set of dated commodities (including money) which have strictly positive prices. For any asymptotic equilibrium and any proper subset of $S(p)$, say $S'(p)$, there exists a pair of commodities, $i(\theta)\epsilon S'(p)$ and $j(v)\epsilon(S(p) - S'(p))$ such that in the solution to the optimization problem for some participant the marginal rate of substitution between $i(\theta)$ and $j(v)$ is defined, is not at a corner solution, and is equal to the price ratio.

This assumption essentially prevents the possibility that (at equilibrium) the economy breaks down into two or more unrelated subeconomies with agents in one using one set of goods and agents in the other using the remainder, each at corner solutions with respect to the goods not used. It is stated as it is because of the possibility that the technologies of firms (or households in their storage activities) can break down into two or more unrelated parts.

Theorem 16.1: The model is globally stable. That is, for any set of initial conditions, there exists a price profile, p^* and a vector of essential stocks, Q^*, defined relative to p^*, such that:

a) (p^*, Q^*) is an asymptotic equilibrium;

b) $p(\theta,t)$ converges weak* to p^*; and

c) $Q(\theta,t)$ converges to Q^*.

Note that this implies that consumption, production, and storage profiles also converge in norm. Indeed, this is evident in the course of the proof.

Proof: In view of Lemmas 15.1 - 15.3, we need only show that every limit point of prices and essential stocks is the same. Suppose instead that there are two different limit points which we call * and **. Then there is a sequence of times at which the economy is very close to * and another sequence at which it is very close to **. In what follows, we choose t_1 as a time from the first sequence and $t_2 > t_1$ as a time from the second.

We first consider households. By strict quasiconcavity of $U(.)$, there exists an $\alpha_1 \geq 0$ such that, for any $\varepsilon_1 > 0$, we can choose t_1 and t_2 large enough to make

$$(16.1) \qquad \int_{t_1}^{\infty} (p(\theta,t_1)\ c(\theta,t_1)\ d\theta \le \int_{t_1}^{\infty} p(\theta,t_1)\ c(\theta,t_2)\ d\theta - \alpha_1 + \varepsilon_1 ,$$

with $\alpha_1 > 0$ unless the limiting consumption profiles, say c* and c**, coincide for all dates and commodities which have a nonzero price at *.[1]

Moreover, as is evident from common sense or from the implication of the first-order conditions given in (5.12), the storage activity of the household must be profit maximizing if we think of it as buying inputs from its consumption activity and selling back outputs all at current market prices. Hence, there exists an $\alpha_2 \geq 0$ such that, for any $\varepsilon_2 > 0$, we can choose t_1 and t_2 large enough to make

$$(16.2) \qquad \int_{t_1}^{\infty} p(\theta,t_1)\ \{y(\theta,t_1) - \phi(y(v,t_1),\ \theta)\}\ d\theta$$

$$\le \int_{t_1}^{\infty} p(\theta,t_1)\ \{y(\theta,t_2) - \phi(y(v,t_2),\ \theta)\}\ d\theta - \alpha_2 + \varepsilon_2 ,$$

where $\alpha_2 > 0$ unless the limiting storage profiles, say y* and y** coincide for all dates and commodities with positive prices at *.

However, * is an asymptotic equilibrium so that the sum of the integrals on the left-hand sides of (16.1) and (16.2) approaches the value of actual holdings at *, while the sum of the integrals on the right-hand sides of the same equations becomes asymptotically no greater than the value of the actual holdings at **, both values being in the prices of *. Thus there exists an $\alpha_3 \geq 0$ such that, for any $\varepsilon_3 > 0$, we can choose t_1 and t_2 large enough to make

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[1] The first-order conditions and the fact that prices are continuous in θ make consumption profiles similarly continuous. Otherwise c* and c** would have to coincide only almost everywhere. A similar remark applies to other optimal profiles below.

$$(16.3) \qquad \int_{t_1}^{\infty} p(\theta,t_1)\, \bar{x}(\theta,t_1)\, d\theta \leq \int_{t_1}^{\infty} p(\theta,t_1)\, \bar{x}(\theta,t_2)\, d\theta - \alpha_3 + \varepsilon_3 ,$$

with $\alpha_3 > 0$ unless c* and c** and y* and y** (and $\bar{x}$* and $\bar{x}$**) coincide as befor

From (5.4),

$$(16.4) \qquad \bar{x}(\theta,t_2) = \bar{x}(\theta,t_1) + \int_{t_1}^{t_2} \bar{r}(\theta,v)\, dv + d(\theta,t_2) - d(\theta,t_1) .$$

Moreover, all profits are paid out in equilibrium (Lemma 13.1); further, since profits are nonincreasing (Theorem 11.1) and bounded below, they approach a limit. Hence $d(\theta,t_2)$ and $d(\theta,t_1)$ approach a common limit. Thus there exists an $\alpha_4 \geq 0$ such that, for any $\varepsilon_4 > 0$, we can choose t_1 and t_2 large enough to make

$$(16.5) \qquad \int_{t_1}^{\infty} p(\theta,t_1) \int_{t_1}^{t_2} \bar{r}(\theta,v)\, dv\, d\theta \geq \alpha_4 - \varepsilon_4 ,$$

with $\alpha_4 > 0$ unless all the household's activities at * and ** coincide for commodities with positive prices at * as before.

Now consider firms. Since the firm's optimal program is profit maximizing,

$$(16.6) \qquad \int_{t_1}^{\infty}\int_{t_1}^{\theta} p(\theta,t_1)\, r(\theta,v,t_1)\, dv\, d\theta$$

$$\leq \int_{t_1}^{\infty}\int_{t_1}^{t_2} p(\theta,t_1)\, \bar{r}(\theta,v)\, dv\, d\theta + \int_{t_1}^{\infty}\int_{t_2}^{\infty} p(\theta,t_1)\, r(\theta,v,t_2)\, dv\, d\theta ,$$

with the strict inequality holding unless the firm as of t_2 is merely carrying out plans made as of t_1 and successfully carried out in the interim, as regards all dates and commodities with a positive price at t_1. Thus, in the limit, the strict inequality will hold unless the firm's limiting stock profiles, say $\bar{x}^*$ and $\bar{x}^{**}$, and production profiles, say y^* and y^{**}, coincide in all dated commodities with positive prices at $*$.[1]

Since $*$ is an asymptotic equilibrium, however, the left-hand side of (16.6) approaches zero, while the second integral on the right-hand side becomes non-positive. Hence, there exists an $\alpha_5 \geq 0$ such that, for any $\varepsilon_5 > 0$, we can choose t_1 and t_2 large enough to make

$$(16.7) \qquad \int_{t_1}^{\infty} p(\theta,t_1) \int_{t_1}^{t_2} \bar{r}(\theta,v)\, dv\, d\theta \geq \alpha_5 - \varepsilon_5 ,$$

---

[1] Purchase profiles need not coincide at t_1 and t_2 (although they do so trivially at $*$ and $**$) because the timing of sales and purchases at given prices is not completely determined by the solution to the firm's optimization problem (and similarly for households).

with $\alpha_5 > 0$ unless the firm's activities at * and ** coincide for positively priced commodities as before.

Now sum (16.5) over households and (16.7) over firms. There exists an $\alpha \geq 0$ such that, for any $\varepsilon > 0$, we can choose t_1 and t_2 large enough to make:

$$(16.8) \qquad \int_{t_1}^{\infty} p(\theta,t_1) \int_{t_1}^{t_2} \{\Sigma_h \bar{r}_h(\theta,v) + \Sigma_f \bar{r}_f(\theta,v)\}\, dv\, d\theta \geq \alpha - \varepsilon ,$$

with $\alpha > 0$ unless for every firm and every household all consumption, storage, production, and stock profiles coincide at * and ** for all positively priced commodities. Since net sales of ordinary commodities by the production sector must be net purchases by the household sector, however, and since actual profits approach a common limit at * and **, the closure equations (7.1) imply that $\alpha > 0$ is impossible.

Thus all profiles coincide for all dated commodities with positive prices at *. It is but a matter of interchanging notation to show that they also coincide for all dated commodities with positive prices at **. It thus remains to show that all positive prices are the same at * as at **.

To do so, observe that the proposition is trivial if money is the only positively priced good at * and **. Suppose, therefore, that there is some other dated commodity with a positive price at one of the equilibria, say *. Then, by the Indecomposability Assumption (Assumption 16.1), there is some dated commodity other than money with a positive price at *, say $i(\theta)$, such that for some participant the marginal rate of substitution between $i(\theta)$ and money is defined, not at a corner solution, and equal to the corresponding price ratio at *. Since all consumption, storage, production, and stock profiles coincide at * and **, however, the marginal rate of substitution in question must be the same at ** as at *, and, since it is not at a corner solution, it must be the case that the price of $i(\theta)$ is the same at ** as at *. If there is now another commodity with a positive price at *, then the Indecomposability Assumption states that there is a commodity with a positive price at *, say $j(v)$, such that the marginal rate of substitution either between $j(v)$ and money or between $j(v)$ and $i(\theta)$ is determined, not at a corner solution, and equal to the corresponding price ratio for some participant. Arguing as before, this means that the price of $j(v)$ must be the same at ** as at *. Evidently we can reach all dated commodities with a positive price at either * or ** in this way; hence all such prices must be the same, and the proof is complete.

17. Incorporating Individual Price Adjustment

It remains to indicate how the analysis can be adapted to the explicit treatment of individual price adjustment along the lines discussed in Section 2 above which, it will be remembered, is important both as a natural justification for the Hahn Process Assumption (Assumption 9.1) and especially for the assumption that price profiles are continuous in dates which allowed us to treat dated commodities at all. Since I have elsewhere treated the issues involved for a simpler model in detail (in [4]) and since they are not essentially different in the present case, I shall merely sketch briefly how this can be done.

The proofs given in the present paper are readily adapted to such a set-up with commodities identified by dealers who set prices allowing others to search over them. Basically, as in Theorems 11.1 and 12.1, target profits and target utilities decrease out of equilibrium so long as the set of dealers with whom a particular agent does business does not change. Since an agent will do business with that set of dealers who offer the most advantageous prices, however, we can think of target profits or target utility as being maximized over the set of dealers. It follows that they still decrease out of equilibrium, even though time derivatives may fail to exist at moments at which the optimum dealer set changes.

The remaining problem occurs because of the possibility that the same price may not be charged for the same commodity by all dealers in equilibrium -- a natural consequence of treating each dealer as defining a separate market. If this is due to persistent advantages on the part of some dealer, such as location, then it is sensible to treat different dealers as really dealing in different commodities and ending up with different prices. If this is not the case, then something needs to be done, not only because the outcome seems unreasonable otherwise, but also because the proof of global stability (Theorem 16.1) involves at a crucial place (the summation of (16.5) and (16.7) to obtain (16.8)) the assumption that everyone deals at the same price for the same commodity (in equilibrium).

One way of accomplishing the desired result is to assume that the *excess* demand facing a dealer who offers a low price for a commodity is greater than that facing a dealer who offers a higher one (at least in equilibrium). This is plausible as regards customer behavior and demand, if information is reasonably good, but requires a little more since it involves excess demand. Essentially what is involved is ruling out the possibility that dealers who happen to offer relatively small amounts can get away with relatively high prices because the demands facing them just happens to be relatively low by the same amount as their supplies, i.e., that the market happens to be so imperfect in a particular way as to permit a very special kind of negative correlation to persist between dealers' prices and the supplies they wish to offer.

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